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# INVERSE FEL PROTON ACCELERATOR VIA PERIODICALLY MODULATED CRYSTAL STRUCTURE

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Presented study explores the idea of using a visible light wave to accelerate relativistic protons via the inverse FEL mechanism. Here, a strain modulated crystal structure – the superlattice, plays the role of a microscopic undulator providing very strong ponderomotive coupling between the beam and the light wave. Purely classical treatment of relativistic protons channeling through a superlattice is performed in a self-consistent fashion involving the Maxwell wave equation for the accelerating electromagnetic field and the relativistic Boltzmann equation for the protons. It yields the accelerating efficiency in terms of the negative gain coefficient for the amplitude of the electromagnetic wave – the rate the energy is extracted from the light by the beam. Presented analytic formalism allows one to find the acceleration rate in a simple closed form, which is further evaluated for a model beam – optical cavity system to verify feasibility of our scheme.

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## 1. INTRODUCTION

Here we suggest using a solid state superlattice as an undulator in conjunction with an optical pumping cavity to accelerate relativistic protons. Heavier particles (vs electrons or positrons) are more appropriate, because they are not susceptible to emitting photons via spontaneous synchrotron radiation and therefore they are more likely to absorb energy from an electromagnetic wave. The idea of the inverse FEL mechanism employed to accelerate charged particles<sup>1</sup> is not new by itself; what we propose here is to replace a conventional magnetic undulator with a microscopically modulated crystal structure – the superlattice, which assures much stronger ponderomotive coupling to the pumping wave. Furthermore, its microscopic undulator periodicity shifts the wavelength of the pumping optical mode to much shorter waves – it opens a possibility of accelerating the beam with high power laser in the visible region.

Such periodic crystal structures occur naturally in several alloy systems or may be prepared artificially with vapor deposition techniques<sup>2</sup>. We consider a superlattice with an accompanying strain modulation, which is a natural consequence of the two constituents having slightly different lattice spacings (eg. Si–Ge superlattice).

The main idea of using a modulated crystal structure as an undulator is illustrated schematically in Figure 1. A beam of relativistic particles while channeling through the crystal follows a well defined trajectory. Presented treatment will be limited to protons but could be modified for negative ions eg.  $H^-$ , as well (providing that the stripping probability for  $H^-$  is not too high). Figure 1 depicts channeling paths (which for protons would lie in low density regions of the crystal and conversely for  $H^-$ ). There are two distinct channeling directions: parallel to the superlattice growth direction and also at a  $45^\circ$  angle to this direction. The latter are what interest us here because, as one can see from Figure 1, the center of the channeling axis is modulated by the superlattice periodicity. This is the essence of the solid state undulator: i.e., the particles are periodically accelerated perpendicular to their flight path as they traverse the channel. The

undulator wavelengths typically fall in the range 50–500 Å, far shorter than those of any macroscopic undulator. Furthermore, the electrostatic crystal-fields involve the line averaged nuclear field and can be two or more orders of magnitude larger than the equivalent fields of macroscopic magnetic undulators (translated into the corresponding electric field in the rest-frame of a relativistic particle). Both of these factors hold the promise of greatly enhanced coupling between the beam and the pumping electromagnetic wave.

## 2. CHANNELING OF RELATIVISTIC PROTONS TROUGH A STRAIN-MODULATED SUPERLATTICE

Following the spirit of the Vlasov equation<sup>3</sup> we will describe a high intensity proton beam in terms of a classical distribution function,  $f(\mathbf{p}, \mathbf{r}, t)$ , governed by the relativistic Boltzmann equation. The transverse dynamics of relativistic protons propagating in a strain-modulated superlattice is modeled by a harmonic crystal field potential<sup>4</sup> and leads to generation of a transverse current. This couples the Vlasov equation to the Maxwell wave equation. Therefore, presented problem reduces to a self consistent solution of the Vlasov and the wave equations, which will be treated in detail in the next few sections. Finally, a closed analytic expression for the amplitude–gain/loss coefficient will be obtained in the linear approximation for appropriate regime with respect to the cavity length and the momentum spread in the incident beam.

We start with a relativistic Lagrangian describing motion of a proton in an arbitrary electromagnetic field  $(\mathbf{A}, \phi)$

$$L = -mc^2 \sqrt{1 - v^2/c^2} + \frac{e}{c} \mathbf{v} \cdot \mathbf{A} - e\phi . \quad (2.1)$$

Here  $\mathbf{A}$  is a vector potential of an electromagnetic field and  $\phi$  is a phenomenological harmonic crystal-field potential, which describes both transverse focusing of the beam and longitudinal modulation of the minimum of the harmonic potential well. More realistic description of the planar channeling is usually given in terms of the Moliere potential<sup>5</sup>, which accounts for averaged electric field of ions and electron cloud around each lattice site. For the purpose of our model calculation a harmonic approximation to the focusing potential is quite sufficient and it can be written as follows

$$\phi = \phi_0 + \frac{1}{2} \phi_1 (x - x_1 \cos gz)^2, \quad (2.2)$$

where  $g = 2\pi/\ell$  is the strain modulation periodicity and  $x_1, \phi_1, \phi_0$  are parameters of the potential. In order to avoid unnecessary complexity we choose to work directly in the laboratory reference frame, and use a space-time description of the problem.

We introduce the canonical momentum of a particle

$$p^\alpha = \frac{\partial L}{\partial \dot{x}^\alpha} = m\gamma v^\alpha + \frac{e}{c} A^\alpha \quad (2.3)$$

and write the Hamiltonian as

$$H = p^\alpha \dot{x}^\alpha - L = m\gamma c^2 + e\phi, \quad (2.4)$$

with

$$\gamma(\mathbf{p}) = \sqrt{1 + \frac{1}{(mc)^2} \left( \mathbf{p} - \frac{e}{c} \mathbf{A} \right)^2}. \quad (2.5)$$

Hamilton's equations of motion are then

$$\dot{x}^\alpha = \frac{\partial H}{\partial p^\alpha} = \frac{1}{m\gamma} \left( p^\alpha - \frac{e}{c} A^\alpha \right), \quad (2.6)$$

$$\dot{p}^\alpha = -\frac{\partial H}{\partial x^\alpha} = \frac{e}{m\gamma} \left( p^\beta - \frac{e}{c} A^\beta \right) \frac{\partial A^\beta}{\partial x^\alpha} - e \frac{\partial \phi}{\partial x^\alpha}. \quad (2.7)$$

A beam of protons moving along the  $z$  axis can be described in terms of a distribution function,  $f(\mathbf{p}, \mathbf{x}, t)$ ; this distribution function obeys the relativistic Vlasov equation

$$\frac{\partial f}{\partial t} + \frac{1}{m\gamma} \left( p^\alpha - \frac{e}{c} A^\alpha \right) \frac{\partial f}{\partial x^\alpha} \quad (2.8)$$

$$+ \left[ \frac{e}{mc\gamma} \left( p^\beta - \frac{e}{c} A^\beta \right) \frac{\partial A^\beta}{\partial x^\alpha} - e \frac{\partial \phi}{\partial x^\alpha} \right] \frac{\partial f}{\partial p^\alpha} = I(p^\alpha) .$$

Here  $I(p^\alpha)$  is a collision integral accounting for various incoherent proton scattering processes. This equation will be treated iteratively and only linear terms in the A-field will be retained. In the 0-th order solution  $A = 0$ , and the corresponding distribution function  $f = f^{(0)}$  is obtained (in the absence of collisions) from the solution of

$$\frac{\partial f^{(0)}}{\partial t} + \frac{p^\alpha}{m\gamma} \frac{\partial f^{(0)}}{\partial x^\alpha} - e \frac{\partial \phi}{\partial x^\alpha} \frac{\partial f^{(0)}}{\partial p^\alpha} = 0 . \quad (2.9)$$

A class of solutions,  $f^{(0)}$ , describing a beam of protons with a sharply peaked initial momentum distribution,  $\Lambda(p_z - p_0)$ ,  $p_x = 0$  can be easily constructed by solving the following set of equations

$$\dot{p}_x = -e \phi_1 (x - x_1 \cos gz), \quad (2.10)$$

$$\dot{p}_z = -e \phi_1 (x - x_1 \cos gz) x_1 g \sin gz . \quad (2.11)$$

Keeping only linear terms in  $x_1$ , the solution has the following form

$$x = \frac{x_1}{1 - U^2} \cos gz, \quad (2.12)$$

and

$$\dot{p}_z = 0 , \quad (2.13)$$

where

$$U = \frac{gp_z}{\sqrt{e\phi_1 m\gamma}} . \quad (2.14)$$

It follows from Eqs.(2.12)–(2.14) that the initial distribution of canonical momenta in the  $z$  direction is conserved during the channeling, while the particles follow the trajectory defined by  $x^{(0)}(z) = x$ . This implies that particles which enter the crystal with  $x$  coordinates off the channel plane lose energy via scattering mechanisms and asymptotically approach the channel plane<sup>6</sup>. Therefore, the solution,  $f^{(0)}$ , can be factorized as follows

$$f^{(0)} = n_0 \delta(x - x^{(0)}) \delta(p_x - p_x^{(0)}) \Lambda(p_z - p_0), \quad (2.15)$$

where

$$p_x^{(0)} = p_z \frac{\partial x^{(0)}}{\partial z} = -p_z x^{(0)} \frac{gx_1}{1 - U^2} \sin gz . \quad (2.16)$$

Here  $\Lambda(p_z - p_0)$  describes an initial momentum distribution and  $n_0$  is a concentration of particles per unit area of the channeling plane. The distribution function,  $f^{(1)}$ , generated by the A-field obeys the following linearized Vlasov equation

$$\begin{aligned} \frac{\partial f^{(1)}}{\partial t} + \frac{p_x}{m\gamma} \frac{\partial f^{(1)}}{\partial x} - \frac{1}{m\gamma} \frac{e}{c} A_x \frac{\partial f^{(0)}}{\partial x} + \frac{p_z}{m\gamma} \frac{\partial f^{(1)}}{\partial z} \\ + \frac{ep_x}{mc\gamma} \frac{\partial A_x}{\partial z} \frac{\partial f^{(1)}}{\partial p_x} - e \frac{\partial \phi}{\partial x} \frac{\partial f^{(1)}}{\partial p_x} = \frac{f^{(1)}}{\tau} . \end{aligned} \quad (2.17)$$

Here we have used a relaxation time approximation for the collision integral modelling it by the relaxation time,  $\tau$ . We have also assumed that only the transverse component of the A-field is present and  $A_x \equiv A(z, t)$ . We seek a solution,  $f^{(1)}$ , in the following form

$$f^{(1)} = n_0 \delta(x - x^{(0)}) \delta(p_x - p_x^{(0)}) h(z, p_z, t) , \quad (2.18)$$

where  $h$  describes bunching of particles due to the presence of the A-field. Substituting Eqs.(2.15)–(2.16) and (2.18) into Eq.(2.17) leads to the following kinetic equation for  $h$

$$\frac{\partial h}{\partial t} + \frac{p_z}{m\gamma} \frac{\partial h}{\partial z} + \frac{e}{c} \frac{p_z}{m\gamma} \frac{g x_1}{1 - U^2} \frac{\partial A}{\partial z} \frac{\partial \Lambda}{\partial p_z} \sin gz = - \frac{h}{\tau} . \quad (2.19)$$

The inhomogeneous term in the above equation plays the role of a driving force representing acceleration of the particles by the ponderomotive force due to the transverse motion (induced by the crystal field) in the presence of the A-field . The transverse current induced by the fields is given by

$$j_x = e \int_{-\infty}^{\infty} dp_x \int_{-\infty}^{\infty} dp_z v_x f^{(1)} , \quad (2.20)$$

where

$$v_x = \frac{1}{m\gamma} \left( p_x - \frac{e}{c} A \right) . \quad (2.21)$$

The current in a single channel can be written to linear accuracy in  $A$  as follows:

$$j_x = - n_0 e \delta(x - x^{(0)}) \int_{-\infty}^{\infty} dp_z h(z, p_z, t) \frac{p_z}{m\gamma} \frac{g x_1}{1 - U^2} \sin gz . \quad (2.22)$$

The net transverse current in the crystal is given by a sum of discrete planar currents concentrated at each channeling site of the  $x$  direction. According to the Appendix, an infinite array of equally spaced currents is equivalent to a smoothed uniform current distribution given by Eq.(2.22) but without the factor  $\delta(x - x^{(0)})$  and with an associated proton density redefined as  $n = n_0/a$ , where  $a$  is the spacing between adjacent

channels. A more general situation where the crystal field potential in each channel is shifted in phase is also discussed in the Appendix. The resulting transverse current couples Eq.(2.19) to the following wave equation

$$\left( \frac{\partial^2}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) A = \frac{4\pi ne}{c} \int_{-\infty}^{\infty} dp_z h \frac{p_z}{m\gamma} \frac{gx_1}{1-U^2} \sin gz, \quad (2.23)$$

resulting in a closed system of equations for  $h$  and  $A$ . Here the  $A$ -field can be identified as a sum of the macroscopic driving field and a self consistent electromagnetic field propagating in the crystal structure. We will not attempt to solve this system of partial-differential-integral equations exactly. For the purpose of calculating a linear amplitude-loss/gain coefficient, it is sufficient to confine the solution for the  $A$ -field to only the first step of the iteration procedure. The procedure is analogous to the Born approximation in scattering theory<sup>7</sup>. We start with a single plane wave solution of arbitrary  $\omega$  and  $k$  propagating in free space along the  $z$  axis in both directions and use it as a 0-th order iteration step

$$A_{\pm}^{(0)} = A_0 e^{-i\omega t \pm ikz}. \quad (2.24)$$

Putting  $A = A_{\pm}^{(0)}$  in Eq.(2.19), one can solve it analytically for  $h = h_{\pm}^{(1)}$  by constructing a Green's function with the appropriate boundary conditions built in it. We consider a finite crystal undulator extending along the  $z$  axis from 0 to  $L$ . The bunching function,  $h_{+}^{(1)}$ , corresponding to the right propagating initial wave,  $A_{+}^{(0)}$ , vanishes outside the undulator and it is assumed to be continuous at the entry point,  $z = 0$ , of the  $A_{+}^{(0)}$  wave. Similarly for the left going solution,  $A_{-}^{(0)}$ , the analogous initial condition is satisfied at its entry point,  $z = L$ . The complete set of boundary conditions can be written in the following compact notation

$$h_{\pm}^{(1)}(z_{\pm}) = 0, \quad (2.25)$$

where

$$z_+ = 0 \quad , \quad z_- = L \quad . \quad (2.26)$$

Finally, the solution of Eq.(2.19) with the above boundary condition (2.25)–(2.26) is expressed in the following integral form

$$h_{\pm}^{(1)}(z) = \int_{z_{\pm}}^z dz' H_{\pm}^{(1)}(z') e^{i(z-z')m\gamma\omega/p_z} \quad , \quad (2.27)$$

where

$$H_{\pm}^{(1)}(z) = \pm ik A_{\pm}^{(0)}(z) \sin gz Q \frac{\partial \Lambda}{\partial p_z} \quad (2.28)$$

and

$$Q = \frac{e}{c} \frac{g x_1}{1 - U^2} \quad , \quad (2.29)$$

where in what follows, collisions are modeled by writing  $\omega \rightarrow \omega + i/\tau$ . Substituting the above solution for  $h_{\pm}^{(1)}$  in Eq.(2.23) reduces it to an inhomogeneous Helmholtz equation for  $A_{\pm}^{(0)}$ :

$$\left( \frac{\partial^2}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) A_{\pm}^{(1)}(z) = J_{\pm}(z) \quad , \quad (2.30)$$

where the explicit formula for  $J_{\pm}$  can be obtained from Eq.(2.27) and Eqs.(2.27)–(2.29). To solve Eq.(2.30) one may integrate it with the following boundary conditions at the entry points for the right and left propagating solutions

$$A_{\pm}^{(0)}(z_{\pm}) = A_{\pm}^{(1)}(z_{\pm}) \quad . \quad (2.31)$$

and

$$\frac{\partial}{\partial z} A_{\pm}^{(0)}|_{z=z_{\pm}} = \frac{\partial}{\partial z} A_{\pm}^{(1)}|_{z=z_{\pm}}. \quad (2.32)$$

The solution can be written explicitly in terms of the Green's function for the Helmholtz equation

$$A_{\pm}^{(1)}(z) = A_{\pm}^{(0)}(z) + \int_{z_{\pm}}^{\frac{z}{\mp}} dz' \frac{e^{ik|z-z'|}}{2ki} J_{\pm}(z') \quad , \quad k = \omega/c \quad , \quad (2.33)$$

where

$$J_{\pm}(z) = \pm ik \int_{-\infty}^{\infty} dp_z F(p_z) \sin gz \int_{z_{\pm}}^z dz' \sin gz' A_{\pm}^{(0)}(z') e^{i(z-z')m\gamma\omega/p_z} \quad , \quad (2.34)$$

with the kernel  $F(p_z)$  defined as follows

$$F(p_z) = 4\pi n Q^2 \frac{p_z}{m\gamma} \frac{\partial \Lambda}{\partial p_z} \quad . \quad (2.35)$$

Carrying out the integration in Eq.(2.34) leads to the required solutions describing the evolution of the right and left propagating waves. To introduce a single pass loss/gain coefficient one should examine the resulting amplitudes after passing through the undulator of length  $L$ , i.e., we have to evaluate  $A_{\pm}^{(0)}(z = z_{\mp})$  by using Eq.(2.34). On the other hand, one can model the effect of coupling by adding a small complex part  $i\kappa^{\pm} = \alpha^{\pm} + i\beta^{\pm}$  to the  $k$ -vector; here  $\alpha$  is a gain/loss coefficient and  $\beta$  describes a small shift in the phase velocity of the optical mode. This could be summarized by the following expression

$$A_{\pm}^{(1)}(z) = A_0 e^{-i\alpha z \pm ikz \pm i\kappa^{\pm}(z-z_{\pm})} \quad . \quad (2.36)$$

Expanding to linear order in  $\kappa^{\pm}$  we obtain

$$A_{\pm}^{(1)}(z) = A_{\pm}^{(0)}(z) \left[ 1 \pm i\kappa^{\pm}(z - z_{\pm}) \right] . \quad (2.37)$$

The above expression evaluated at  $z = z_{\pm}$  simplifies to

$$A_{\pm}^{(1)}(z_{\mp}) = A_{\pm}^{(0)}(z_{\mp}) (1 \pm i\kappa^{\pm}L) . \quad (2.38)$$

Further comparison of Eqs.(2.38) and (2.34) (with  $z = z_{\mp}$ ) allows one to identify two complex loss/gain coefficients  $\kappa^{\pm}$ , as follows

$$\kappa^{+} = \frac{1}{2Li} \int_{-\infty}^{\infty} dp_z F(p_z) \int_0^L dz' e^{-ikz'} \sin gz' \int_0^{z'} dz'' \sin gz'' e^{i(z' - z'')m\gamma\omega/p_z + ikz''} \quad (2.39)$$

and

$$\kappa^{-} = \frac{1}{2Li} \int_{-\infty}^{\infty} dp_z F(p_z) \int_0^L dz' e^{-ikz'} \sin gz' \int_{z'}^L dz'' \sin gz'' e^{i(z' - z'')m\gamma\omega/p_z + ikz''} . \quad (2.40)$$

Apart from a simple integration over  $z'$  and  $z''$ , we have found the complex coefficient,  $\kappa^{\pm}$ , for waves propagating parallel and antiparallel to the particle beam. The imaginary part of  $\kappa^{\pm}$  describes either spontaneous amplification or degradation (depending on its sign) of the optical mode. In the next section, the above integrations will be carried out explicitly and the linear gain/loss coefficients will be calculated in a closed form.

### 3. LINEAR LOSS/GAIN COEFFICIENT – ACCELERATION RATE

On carrying out the two spatial integrations in Eqs.(2.39) and (2.40), the expressions for the complex gain,  $\kappa^\pm$ , reduce to

$$\begin{aligned} \kappa^+ = \frac{L}{16} \int_{-\infty}^{\infty} dp_z F(p_z) \left\{ \frac{1}{\mu^-} [N(\mu^+) - N(\mu^-) - N(2g) - N(0)] \right. \\ \left. + \frac{1}{\mu^+} [N(\mu^-) - N(\mu^+) + N(0) - N(-2g)] \right. \\ \left. - e^{-2ikL} \frac{1}{\mu^-} [N(v^-) - N(v^+) + N(2k + 2g) - N(-2k)] \right. \\ \left. - e^{-2ikL} \frac{1}{\mu^+} [N(v^-) - N(v^+) + N(2k) - N(2k - 2g)] \right\} \end{aligned} \quad (3.1)$$

and

$$\begin{aligned} \kappa^- = \frac{L}{16} \int_{-\infty}^{\infty} dp_z F(p_z) \left\{ \frac{1}{v^-} [N(2g) - N(0) - N(2g - 2k) + N(-2k)] \right. \\ \left. - \frac{1}{v^+} [N(0) - N(-2g) - N(-2k) + N(-2k - 2g)] \right. \\ \left. - e^{-iv^-L} \frac{1}{v^-} [N(v^+) - N(v^-) - N(\mu^+) + N(\mu^-)] \right. \\ \left. + e^{-iv^+L} \frac{1}{v^+} [N(v^+) - N(v^-) - N(\mu^+) + N(\mu^-)] \right\} , \end{aligned} \quad (3.2)$$

where

$$\mu^\pm = m\gamma\omega/p_z - k \pm g \quad , \quad v^\pm = m\gamma\omega/p_z + k \pm g \quad (3.3)$$

and

$$N(x) = \frac{e^{ixL/2}}{L/2} \frac{\sin(xL/2)}{xL/2} . \quad (3.4)$$

Using our definition of  $\kappa^\pm$  the linear gain coefficient is given by  $\alpha^\pm = -\text{Im } \kappa^\pm$  we observe that the function

$$\Gamma(x) \equiv \text{Im } \frac{N(x)}{x} = \left( \frac{\sin(xL/2)}{xL/2} \right)^2 \quad (3.5)$$

is the characteristic form occurring in diffraction theory, with the principal maximum at  $x = 0$ . The remaining terms in curly brackets in Eqs.(3.1) and (3.2) do not have this feature and their contribution can be neglected relative to the terms containing  $Q$ . Therefore, Eqs.(3.1) and (3.2) simplify to

$$\alpha^+ = \frac{L}{16} \int_{-\infty}^{\infty} dp_z F(p_z) [\Gamma(\mu^+) + \Gamma(\mu^-)] \quad (3.6)$$

and

$$\alpha^- = -\frac{L}{16} \int_{-\infty}^{\infty} dp_z F(p_z) [\Gamma(v^+) + \Gamma(v^-)] . \quad (3.7)$$

Since the quantities  $g$ ,  $k$  and  $p_z$  are positive, for the chosen geometry, only  $\mu^-$  and  $v^-$  pass through zero. Therefore, the remaining terms,  $\Gamma(\mu^+)$  and  $\Gamma(v^+)$  can be neglected when evaluated far from the maximum compared to the  $\Gamma(x) = 1$ , ( $x = 0$ ) term. Using the above argument Eqs.(3.6) and (3.7) become

$$\alpha^+ = \frac{\pi}{4} n \int_{-\infty}^{\infty} dp_z Q^2 \frac{p_z}{m\gamma} \frac{\partial \Lambda}{\partial p_z} L\Gamma(\mu^-) \quad (3.8)$$

and

$$\alpha^- = -\frac{\pi}{4} n \int_{-\infty}^{\infty} dp_z Q^2 \frac{p_z}{m\gamma} \frac{\partial \Lambda}{\partial p_z} L\Gamma(v^-) , \quad (3.9)$$

where the explicit form of the kernel  $F$  is used.

Imposing resonant condition,  $v^- = 0$ , in Eq.(3.9) fixes the wavevector of the optical mode as follows

$$g = m\gamma kc/p_z + k \quad (3.10)$$

where

$$p_z = m\gamma\beta c . \quad (3.11)$$

Assuming the extreme relativistic limit,  $\beta \rightarrow 1$ , for the proton beam yields the following resonance condition

$$\lambda = 2\ell . \quad (3.12)$$

One can summarize the above condition by the following statement: pumping the beam with the electromagnetic wave of twice the superlattice period would result in the maximum rate of energy extraction from the wave, given here by  $\alpha^-$ , which is the essence of the presented accelerating scheme.

One can notice in passing, that  $\alpha^+$  corresponds to the lasing process, when the forward propagating wave is being amplified by the energy drawn from the beam (the FEL effect). The corresponding resonant condition is given by  $\mu^- = 0$ , which yields a well known scaling relation between the undulator period and the amplified wavelength

$$\lambda = \frac{\ell}{2\gamma^2} . \quad (3.13)$$

From now on, we are interested only in the inverse FEL situation, therefore we will simplify further notation by putting:  $\alpha \equiv \alpha^-$  and  $v \equiv v^-$ .

One can notice that, apart from a slowly varying function  $F$ , the remaining functions occurring in the integrand in Eq.(3.9), namely,  $\Lambda$  and  $\Gamma$  are sharply peaked functions of momentum characterized by the respective widths  $\left(\frac{\Delta p}{p}\right)_\Lambda$  and  $\left(\frac{\Delta p}{p}\right)_\Gamma$ . Here the width of the initial momentum distribution,  $\Lambda$ , was introduced in a standard way<sup>8</sup>

$$\left(\frac{\Delta p}{p}\right)_\Lambda = \left[ \left( p_z^2 \frac{\partial \Lambda}{\partial p_z} \right)_{\max} \right]^{-1/2} . \quad (3.14)$$

Similarly the width of  $\Gamma$  is governed by the following simple ratio

$$\left(\frac{\Delta p}{p}\right)_\Gamma = \frac{l}{L} . \quad (3.15)$$

Now one can compare relative sharpness of both functions;  $\Lambda$  and  $\Gamma$ . Typical value of the relative momentum spread is of the order of  $10^{-4}$ . Assuming superlattice modulation of  $500\text{\AA}$  and crystal length of 5 cm allows one to evaluate the width of  $\Gamma$ . Both characteristic widths can be summarized as follows

$$\left(\frac{\Delta p}{p}\right)_\Lambda = 10^{-4} , \quad \left(\frac{\Delta p}{p}\right)_\Gamma = 10^{-6} . \quad (3.16)$$

The integration in Eq.(3.9) is carried out assuming that the sharper function, namely  $\Gamma$ , is approximated by the  $\delta$ -function according to the following asymptotic relationship

$$\lim_{L \gg l} \frac{L}{2\pi} \Gamma(v) = \delta(v) . \quad (3.17)$$

Applying resonant condition, described by Eqs.(3.10)–(3.12), and assuming  $\beta \rightarrow 1$  reduces the gain/loss coefficient to the following simple expression

$$\alpha = -\frac{\pi}{2} \ln \frac{Q^2}{m^2} \gamma \ell \left( \frac{\Delta p}{p} \right)_\Lambda^{-2}. \quad (3.18)$$

The above final result will serve as a starting point for further feasibility discussion.

#### 4. THREE WAVE MIXING – PHYSICAL PICTURE

According to the presented model calculation spontaneous bunching of the proton beam channeling through a superlattice and interacting with the electromagnetic wave results in energy flow from the wave to the beam. This particular kind of particle density fluctuation,  $h$ , has the form of a propagating plane wave of the same frequency,  $\omega$ , as the emitted electromagnetic wave. The phase velocity of the moving bunch matches the velocity of protons in the beam. Therefore, the quantity  $\gamma m \omega / p_z \equiv k_b$  represents the wavevector of the propagating particle density bunch. Keeping in mind that the periodicity of the undulator represents a static wave with a wavevector  $g$ , and that  $k$  is the wavevector of the electromagnetic wave, we can analyze our results in the language of three wave mixing.

One can notice, that the resonant denominators appearing in final expressions for the linear gain/loss coefficients, Eq.(3.3), i.e.,  $\mu^\pm = m\gamma\omega/p_z - k \pm g$  (forward propagating wave) and  $\nu^\pm = m\gamma\omega/p_z + k \pm g$  (backward propagating wave), can be identified with the frequency – wavevector conservation conditions for a three wave mixing process. From this point of view, the FEL or the inverse FEL reduce to an Umklapp process<sup>8</sup> involving: 1) the propagating "bunch" in the proton density, 2) the pumping electromagnetic wave and 3) the static periodic field of the undulator. Matching of frequencies for the two "dynamic modes" assures "energy" conservation. Furthermore "momentum" conservation of all three modes (static and dynamic) yields the  $\mu^- = 0$  (FEL),  $\nu^- = 0$  (inverse FEL) conditions. The last condition is equivalent to a momentum "recoil",  $g$ , between the particle density "bunch" and the electromagnetic wave, i.e., an Umklapp process. One can see immediately that the resonant denominator,  $\nu^-$ , is responsible for the negative gain (deamplification of the backward propagating wave), where a four momentum  $(0, g)$  is transferred from the backward propagating wave to the forward moving proton bunch.

## 5. FEASIBILITY ASSESSMENT

In this section we will discuss the feasibility of the proposed scheme by considering (110) planar channeling in a strain modulated Si crystal<sup>9</sup>. We write the undulator period as  $\ell = Nd$ , where  $d = 1.92 \text{ \AA}$  is the spacing between successive lattice planes and  $N$  is the number of such planes. The strain modulation, of course, requires a second component, such as Ge; however, we will use the parameters of Si for convenience.

One can identify the magnetic field of the undulator, which would be equivalent to electric field of the strain modulated crystal lattice (i.e., would result in the same transverse velocity of the channeling particles). The equivalence is given by:

$$B = \phi_1 x_1 \quad (5.1)$$

According to Ref. 9, the harmonic part of the crystal field potential for (110) planar channeling in Si is  $e\phi_1 = 36 \text{ eV \AA}^{-2}$ . Assuming a strain modulation amplitude of  $0.1 \text{ \AA}$  leads to an enormous magnetic field of  $B = 1.2 \times 10^6 \text{ Gauss}$ .

Relativistic particles while channeling along the path undergo transverse harmonic oscillations from the crystal field potential, an analog of the betatron oscillations, with the characteristic frequency  $\omega_\beta = \sqrt{e\phi_1/m}$ . One can see from Eq.(2.29) that if the angular velocity of a particle traversing the strain modulated path,  $\omega = 2\pi v_\perp/\ell$ , approaches  $\omega_\beta/\sqrt{\gamma}$  (Doppler shifted betatron frequency), the undulator parameter,  $Q$ , has a resonance ( $U \rightarrow 1$ ), which would enormously enhance the gain/loss coefficient. However, the excessive growth of the undulator parameter would soon result in a rapid dechanneling of the particles. One can see this easily if  $Q$  is rewritten in the following form

$$Q = \frac{e}{c} \frac{v_\perp}{v_\parallel} \quad (5.2)$$

where  $v_{\perp}, v_{\parallel}$  are transverse and longitudinal components of the particle velocity, respectively.

Now, the following simple physical criterion allows one to estimate the maximum value of  $Q$ . Dechanneling will occur if the transverse kinetic energy of the particle exceeds the binding energy of the harmonic potential (a particle leaves the channel). If the maximum transverse velocity of a channeling particle is  $v_{\perp}$  and  $a$  is the distance between adjacent channels (for (110) channeling in Si  $a = 5 \text{ \AA}$ ), the above condition can be written as follows:

$$v_{\perp}^2 \geq \frac{e\phi_1}{m} \left(\frac{a}{2}\right)^2. \quad (5.3)$$

The equality sign in Eq.(5.3) along with Eq.(5.2) fix the maximum allowed value of the undulator parameter as

$$Q^{\max} = \frac{e}{c} \frac{a}{2} \sqrt{\frac{e\phi_1}{mc^2} \frac{\gamma}{\gamma^2 - 1}}. \quad (5.4)$$

The above expression can be evaluated for relativistic protons channeling through our model superlattice as

$$Q^{\max} = 7.5 \times 10^{-24} \text{ cm}^{1/2} \text{ g}^{1/2}. \quad (5.5)$$

Now, one can evaluate Eq.(3.18) assuming only one proton – by assigning  $n$  to be an inverse area of the channeling plane per one particle for typical values of the beam concentration,  $n = 10^{16} \text{ cm}^{-2}$ . This way  $\alpha$  describes the rate of optical amplitude depletion per one particle – the acceleration rate. Assuming  $\gamma$  of 100,  $\ell = 500 \text{ \AA}$  and  $\left(\frac{\Delta p}{p}\right)_{\Lambda} = 10^{-4}$  yields the following value of the acceleration rate

$$\alpha = 2.65 \times 10^{-2} \text{ cm}^{-1}. \quad (5.6)$$

The nominal acceleration efficiency in units of eV/cm will, obviously, depend on the energy density of the actual optical cavity, which is left out for further discussion elsewhere.

## 6. CONCLUSIONS

The final acceleration rate evaluated numerically in Eq.(5.6) seems to be quite substantial. Practical feasibility of the presented scheme rests on availability of high density optical mode pumped by a high power laser. Furthermore, there is some radiation damage and thermal heating of the crystal associated with proton channeling. One can apply the beam to a rapidly spinning crystal<sup>10</sup>, so that the average power density could be reduced to alleviate this problem.

## APPENDIX

We will generalize the treatment given in Section 2. Eqs.(2.19)–(2.24), to a situation depicted schematically in Figure 2. Now there is a constant phase shift,  $\rho$ , between the crystal field of two neighboring channels. In this case the kinetic equation, Eq.(2.19), can be modified as follows

$$\left( \frac{\partial}{\partial t} + v_z \frac{\partial}{\partial z} \right) h^j(z,t) = M \frac{\partial A}{\partial z} \sin(gz + j\rho) . \quad (\text{A.1})$$

Here  $j$  is the index of the channel, which discretizes the variable  $x$  as follows:  $x = aj$ . Furthermore we denote

$$v_z = \frac{p_z}{m\gamma} \quad (\text{A.2})$$

and

$$M = v_z Q \frac{\partial \Lambda}{\partial p_z} , \quad (\text{A.3})$$

Assuming an  $x$ -dependent  $A$ -field of the form

$$A(x,z,t) = A(z) e^{-i\omega t + ik_x x} , \quad (\text{A.4})$$

we seek a solution for  $h$  in the following form

$$h^j(z,t) = h^j(z) e^{-i\omega t} . \quad (\text{A.5})$$

Substituting the above expression in Eq.(A.1) one can rewrite it as follows

$$\left( v_z \frac{\partial}{\partial z} - i\omega \right) h^j(z) = \frac{M}{2i} \left( e^{igz + i(k_x a + \rho)j} - e^{-igz + i(k_x a - \rho)j} \right) \frac{\partial A}{\partial z} . \quad (\text{A.6})$$

Therefore the  $j$ -dependence of our solution can be written explicitly as

$$h^j(z) = h_+(z) e^{i(k_x a + \rho)j} + h_-(z) e^{i(k_x a - \rho)j} , \quad (\text{A.7})$$

where the functions  $h_+$  and  $h_-$  are governed by the following pair of equations (equivalent to Eq.(A.2)

$$\left( v_z \frac{\partial}{\partial z} - i\omega \right) h_+(z) = \frac{M}{2i} e^{igz} \frac{\partial A}{\partial z} , \quad (\text{A.8})$$

and

$$\left( v_z \frac{\partial}{\partial z} - i\omega \right) h_-(z) = -\frac{M}{2i} e^{-igz} \frac{\partial A}{\partial z} . \quad (\text{A.9})$$

The net transverse current in the crystal, induced by  $h_+$  and  $h_-$ , couples to the A-field through the following wave equation

$$\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + k^2 \right) A(x, z) = 4\pi n_0 \int_{-\infty}^{\infty} dp_z v_z Q \sum_j \delta(x - ja) \sin(gz - jp) h^j(z) , \quad (\text{A.10})$$

where

$$k^2 = k_x^2 + k_z^2 = (\omega/c)^2 .$$

Substituting according to Eq.(A.3) and representing the  $\delta$ -function as an integral form,  $\delta(x) = \int_{-\infty}^{\infty} dk_x e^{ik_x x}$ , one can rewrite Eq.(A.5) as follows

$$\left( \frac{\partial^2}{\partial z^2} + k_z^2 \right) A(z) = 4\pi n_0 \int_{-\infty}^{\infty} dp_z v_z Q \frac{1}{2ia} \quad (\text{A.11})$$

$$\times \left( h_- e^{-igz} - h_+ e^{igz} - h_- e^{-igz - i(2\rho/a)x} - h_+ e^{igz + i(2\rho/a)x} \right) .$$

Obviously for a channeling particle  $x \ll a$  and therefore  $e^{i(2\rho/a)x} \cong 1$ ; Eq.(A.7) then becomes

$$\left(\frac{\partial^2}{\partial z^2} + k_z^2\right)A(z) = 4\pi n_0 \int_{-\infty}^{\infty} dp_z v_z Q \frac{1}{a} (h_- + h_+) \sin gz . \quad (\text{A.12})$$

Similarly from Eq.(A.8) one obtains

$$\left(v_z \frac{\partial}{\partial z} - i\omega\right) [h_+(z) + h_-(z)] = M \frac{\partial A}{\partial z} \sin gz . \quad (\text{A.13})$$

One can see immediately that redefining;  $h_+ + h_- = h$ , in Eqs.(A.7) and (A.8) reduces them to the initial pair of equations used previously, namely Eqs.(2.19) and (2.22). Therefore, our case ( $x \ll a$ ) is equivalent to the one treated before in Section 2 and the same final expression for the linear gain/loss holds even in case of finite phase shift in the crystal potential between neighboring channels.

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## FIGURE CAPTIONS

Figure 1 Center of the channeling trajectory for a [110] direction in a strain-modulated superlattice;

a) for protons (located in a low density region),

b) for  $H^-$  (located in a high density region).

The [100] channeling direction yields no undulator effect.

Figure 2 An infinite array of parallel channels in a strain-modulated superlattice. The phase shift in the crystal field potential between two neighboring channels,  $\rho$ , is given by  $\rho = \frac{a\pi}{l}$ .

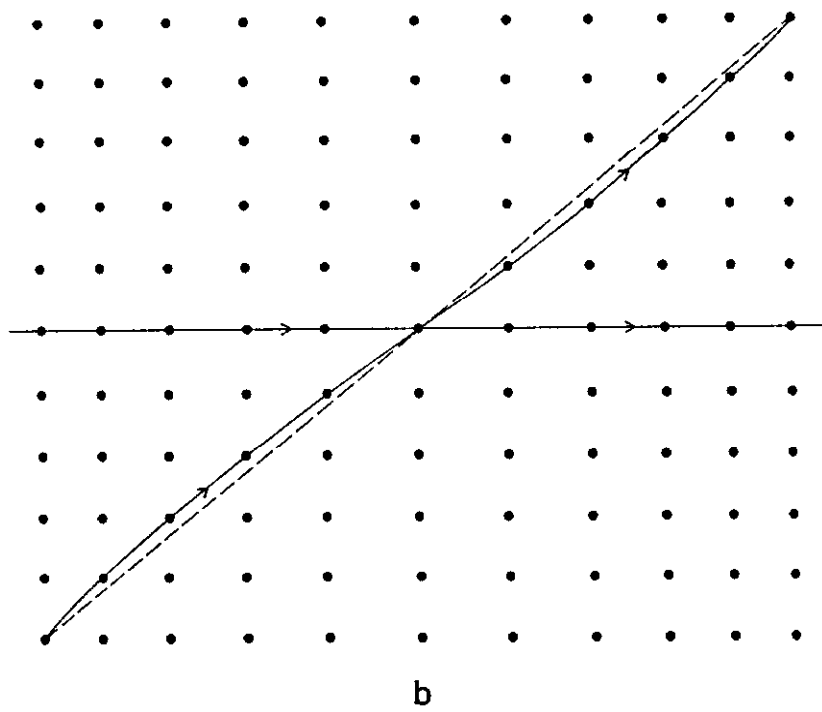
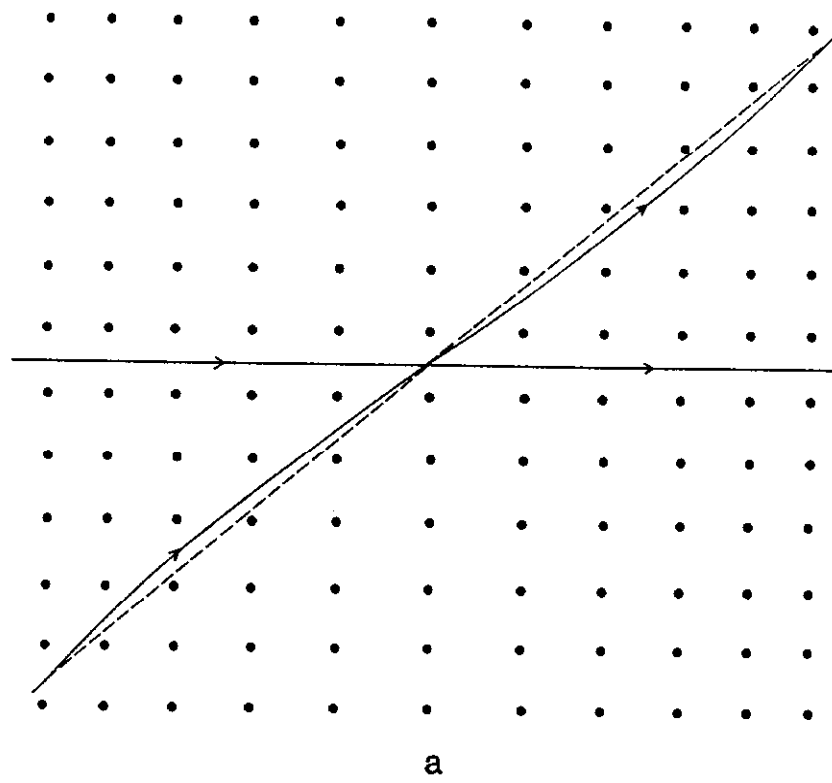


Figure 1

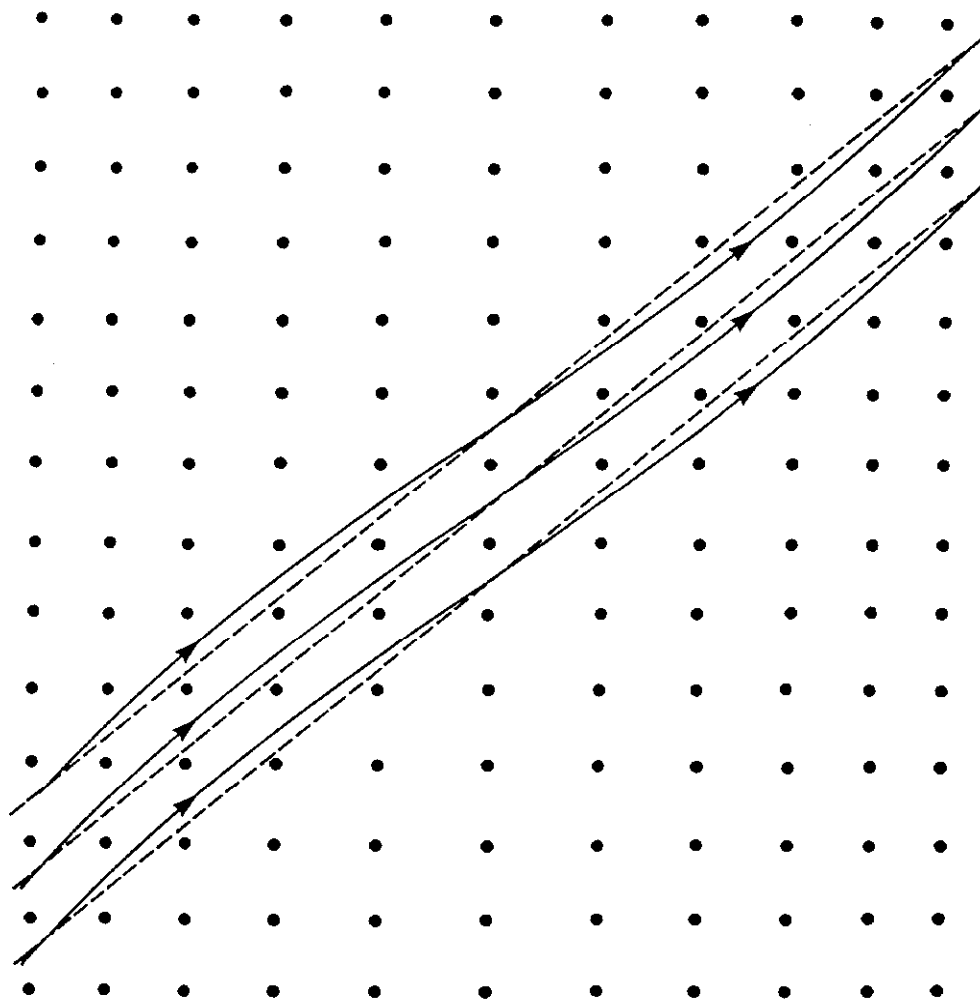


Figure 2